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Logic of Simultaneity

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Abstract A logical model of spatiotemporal structures is pictured as a succession of processes in time. One usual way to formalize time structure is to assume the global existence of time points and then collect some of them to form time intervals of processes. Under this set-theoretic approach, the logic that governs the processes acquires a Boolean structure. However, in a distributed computer system or a relativistic universe where the message-passing time between different locations is not negligible, the logic has no choice but to accept time interval instead of time point as a primitive concept. The resulting logico-algebraic structure matches that of orthologic, which is known as the most simplified version of quantum logic, and the *conventionality of simultaneity* claim is reduced to the non-distributivity of the logic.

Keywords Conventionality of simultaneity · Special relativity · Orthologic · Quantum logic

1 Introduction

1.1 Background

As models of concurrent, parallel and distributed systems, the process algebras are widely studied in computer science [1]. Although the word *process algebra* covers a very broad scope, the following features are shared in common [13]:

- Representing interactions between independent processes as communication (message-passing), rather than as the modification of shared variables.
- Describing processes and systems using a small collection of primitives, and operators for combining those primitives.

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– Defining algebraic laws for the process operators, which allow process expressions to be manipulated using equational reasoning.

Here one might think of those operators as algebraic ones such as sequential composition or parallel composition. Another approach for modeling concurrent processes is to adopt logical operators, whereby properties of a system including mutual exclusion, accessibility, and correctness can be described appropriately. Temporal logic is one of the best known and most widely used formalisms for such purposes [5].

There is a fundamental problem we must face when considering time logically: as is well known to philosophers of science, the concept of simultaneity between spatially separated locations might just be a convention [7, 14, 15]. Malament [11] demonstrates that the standard simultaneity can be determined without convention by assuming that the simultaneity relation to be a certain equivalence relation, while some researchers such as Grünbaum point out that Malament's assumption is too strong [12].

We handle this problem in a different manner. That is, we focus on the logico-algebraic structure of spacetime, and reduce its conventional characteristic to the non-distributivity of logical operations.

Before getting into the details of our formalization, we look at some ontological background concerning this subject.

First, there has been a long controversy over the very definition of time. We prefer to take the view of Reichenbach [15], who defines time using causality relation. As Grünbaum indicates, however, Reichenbach's construction of time based on his mark method contains a vicious circle—he implicitly employs the concept of time in order to describe the causality. Therefore we define our model so as not to distinguish between cause and effect of the causality relation. As a result, the relation turns out to be symmetric, and the generalized simultaneity relation can be defined as the complement relation of the causality relation.

The second controversy is related to the elements of time. In addition to the time point (instant) based approach, various formalisms of interval (period) based time have been studied in the fields of philosophy, linguistics, artificial intelligence, and computer science [4, 6, 8, 9]. Adopting the latter perspective, we use the notion of intervals as the primitives of time.

1.2 General Assumptions

We suppose a logical representation of the physical universe that consists of causal sequences of events occurring at different spatial locations. These locations are here referred to as *sites*. Formally, a site is a sequence of processes ordered by the happened-before relation. The term *process* denotes one of the continuing states in a site, including in particular nothing happening states. It is also assumed for simplicity that (i) the processes in each site occur consecutively with no time gaps, (ii) each process lasts for some non-zero time duration. Any change of processes is referred to as an *event* (Fig. 1).

2 Concept of Time in Non-Relativistic Theories

For the later comparison, this section is devoted to introduce a formalization of time in non-relativistic situations. By the word *non-relativistic universe* we mean a collection of sites on which there is no upper bound on the absolute velocity of information transmission. The earliest such treatment of time is due to Russel [16].

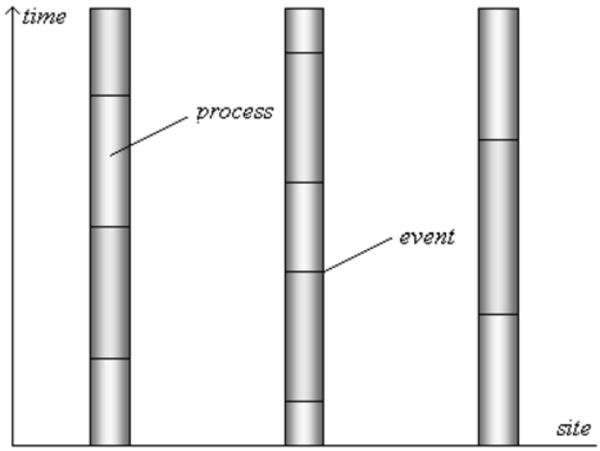


Fig. 1 Processes and events

2.1 Logical Construction of Time

Since we assume that any process is not instantaneous, but occupies some non-zero time duration, we can say that a process p *is earlier than* a process q if p ends before q begins, and that p *is simultaneous with* q if p partly or completely overlaps with q, i.e. neither p is earlier than q nor vice versa. In particular, p is simultaneous with p itself. Note that this relation of simultaneity is reflexive and symmetric, but not transitive.

A maximal set of simultaneity, which is referred to as a *time point*, is then defined as a maximal set of processes (with respect to the set inclusion ordering), any two members of which are simultaneous with each other. The collection of all time points is denoted by $\mathcal{P}$, i.e.

$$\mathcal{P} \equiv \{T \in 2^{\mathbf{Proc}} \mid \forall p \in T.S(p,q) \Leftrightarrow q \in T\} \tag{1}$$

where **Proc** denotes the set of all processes, $2^{\mathbf{Proc}}$ denotes the power set of **Proc** (the set of all subsets of **Proc**), and S denotes the simultaneity relation.

Example 1 In Fig. 2, we have $T_1 = \{p_1, q_1, r_1\}$, $T_2 = \{p_1, q_2, r_1\}$, $T_3 = \{p_2, q_2, r_1\}$, $T_4 = \{p_2, q_2, r_2\}$, $T_5 = \{p_2, q_3, r_2\}$, $T_6 = \{p_3, q_3, r_2\}$, $T_7 = \{p_3, q_4, r_2\}$, $T_8 = \{p_3, q_4, r_3\}$, $T_9 = \{p_4, q_4, r_3\}$, $T_{10} = \{p_4, q_5, r_3\}$. The broken lines represent the simultaneous time points.

It is worth noting that the happened-before relation on **Proc** defines a linear ordering on $\mathcal{P}$. For any distinct time points T and T', there must be a process p that is in T but not in T', and a process p' that is in T' but not in T. By the definition of time point, either of the following holds: "p is earlier than p'" or "p' is earlier than p". In the former case let $T < T'$ and in the latter case $T' < T$.

Time points are used to introduce the concept of *time intervals*. For each process p, its time interval $[p]$ is defined as the set of all time points that contain p, i.e.

$$[p] \equiv \{T \in \mathcal{P} \mid p \in T\} \tag{2}$$

Example 2 In Fig. 2, we see that $T_1 < T_2 < \cdots < T_{10}$, and that $[p_1] = \{T_1, T_2\}$, $[q_1] = \{T_1\}$, $[r_1] = \{T_1, T_2, T_3\}, \ldots$.

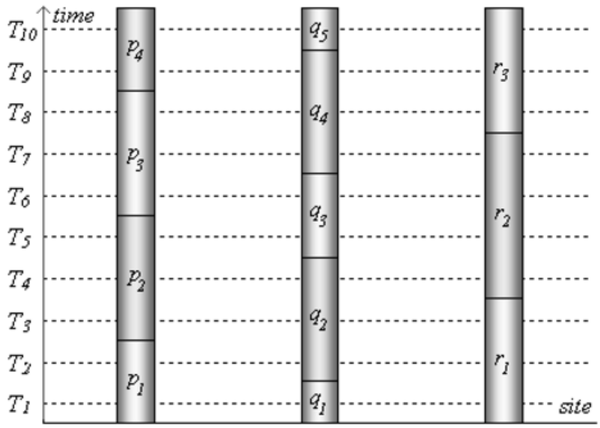


Fig. 2 Time points

2.2 Logic of Simultaneity in Non-Relativistic Theories

We are now led to the logic of simultaneity by considering a process p and its time interval $[p]$ as an atomic *proposition* and its *truth value*, respectively. An atomic proposition p asserts that *the process* p *is occurring*. Complex propositions are built out of atomic propositions and logical connectives:

1. For any proposition p, $\neg p$ denotes the proposition that the proposition p is not true. The truth value of $\neg p$ is defined as $[\neg p] \equiv \mathcal{P} - [p]$ (set-theoretic complement relative to $\mathcal{P}$), which amounts to the time interval that p is not true.
2. For any propositions p and q, $p \wedge q$ denotes the proposition that the propositions p and q are both true. The truth value of $p \wedge q$ is defined as $[p \wedge q] \equiv [p] \cap [q]$ (set-theoretic intersection), which amounts to the time interval that p and q are both true.
3. For any propositions p and q, $p \vee q$ denotes the proposition that at least one of the propositions p and q is true. The truth value of $p \vee q$ is defined as $[p \vee q] \equiv [p] \cup [q]$ (set-theoretic union), which amounts to the time interval that at least one of p and q is true.

Since the operations coincide with the usual set-theoretic ones, it is obvious that the resulting logic is Boolean.

3 Concept of Time in Relativistic Theories

3.1 Undecidability of Simultaneity in Relativistic Theories

In the preceding section, we have implicitly assumed the existence of the global clock, which is represented by the linearly arranged time points. However, the classical concept of simultaneity loses its meaning in a real distributed system or a relativistic universe. What the principle of special relativity says is that it does take a non-zero time duration to transmit any causal signals between spatially separated sites (for a basic reference, see [17]). Accordingly, by the word *relativistic universe* we mean a collection of sites on which there is an upper bound on the absolute velocity of information transmission. As shown in Fig. 3, we refer to any signal capable of transmitting information between sites as a *message*.

Fig. 3 Messages between sites

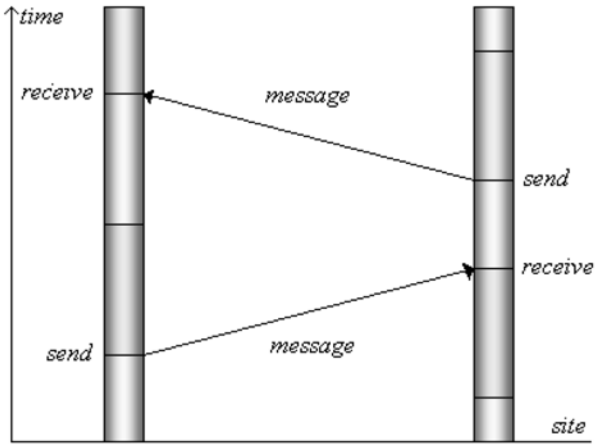
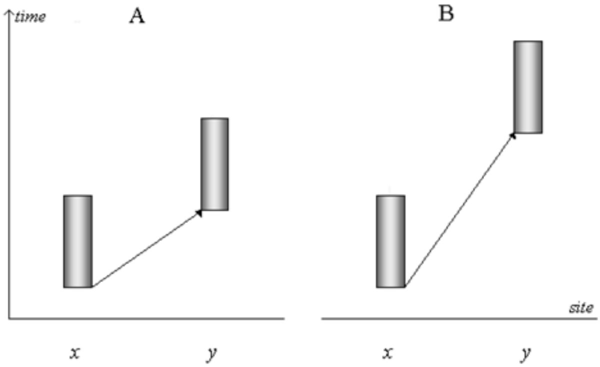


Fig. 4 Indiscernible situations in a relativistic universe



While a message transmission is needed to synchronize two clocks at different sites, the transmission time duration is not measurable without reference to synchronized clocks located at both sites; it is a vicious circle. Hence we must abandon the attempt to provide the global time points, i.e. the clock common to all sites. Thus the simultaneity based on the overlap relation cannot be defined since it cannot generally be determined whether two processes at different sites overlap temporally.

Example 3 In a relativistic universe, it is essentially meaningless to distinguish between the figures A (depicted as x and y are overlapping) and B (depicted as x and y are not overlapping) (Fig. 4).

3.2 Temporal Containment

We should therefore focus on the special cases where we can say with certainty that two or more processes run simultaneously. In simple cases where a message is sent from a site x to a site y and then another message is sent back from y to x, it is verifiable that the processes in x that occur consecutively with no time gaps between the sending event and the receiving event *temporally contain* the processes in y that occur between the receiving event and the sending event.

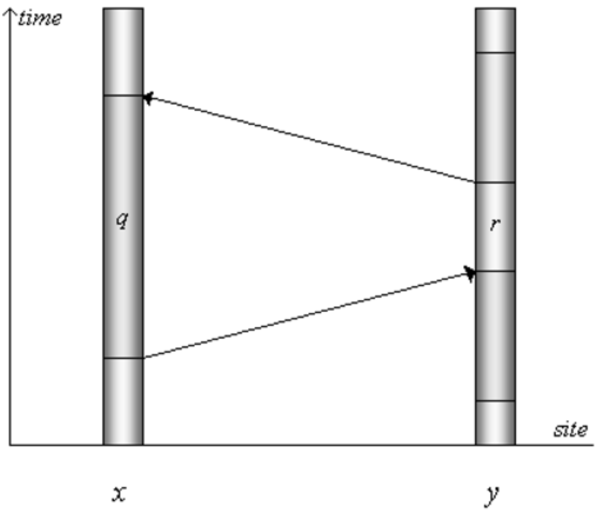


Fig. 5 Temporal containment

Example 4 Temporal containment: a process q is occurring whenever a process r is occurring (Fig. 5).

In other words, the processes like q and r shown in Fig. 5, which are in the relation of temporal containment, can be said to be simultaneous even in the relativistic universe. In the following discussion, we drop the idea of defining time interval via simultaneous time points, but directly construct the logic employing temporal containment relation between processes.

3.3 Logic of Simultaneity in Relativistic Theories

We say that two processes have a *causal relationship* in a relativistic universe if they are linked with the happened-before relation [10]. The happened-before relation B on **Proc** is defined as the smallest relation satisfying the following conditions: (i) If p and q are processes of the same site, and p occurs before q, then $B(p,q)$. (ii) If p ends with an event of sending a message and q begins with an event of receiving that message, then $B(p,q)$. (iii) If $B(p,q)$ and $B(q,r)$, then $B(p,r)$. The causality relation C on **Proc** is then defined as the smallest relation satisfying the following condition: $C(p,q)$ if and only if $B(p,q)$ or $B(q,p)$. Since B is irreflexive, and C is the symmetrized relation of B, C is irreflexive and symmetric, but no longer transitive.

Using the same notation as before, we informally denote by $[p]$ the time interval of a process p. The example shown in Fig. 5 thus indicates that $[q] \supseteq [r]$. This containment relation is characterized by the fact that *any process that has a causal relationship with q has a causal relationship with r*, i.e.

$$\forall p.(C(p,q) \Rightarrow C(p,r)). \tag{3}$$

To formalize a more general setting where a process is covered by two or more processes (Fig. 6), we need a slight modification of the formula (3). Letting $[q_1, q_2, \ldots]$ be the time interval that at least one of $\{q_1, q_2, \ldots\}$ occurs, we say that $[q_1, q_2, \ldots] \supseteq [r]$ if *any process*

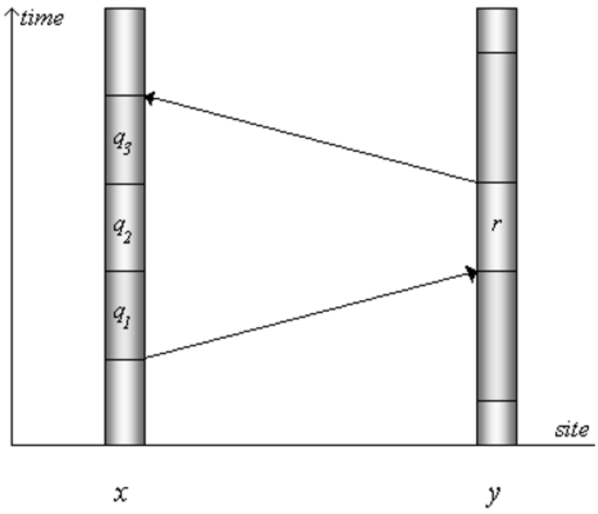


Fig. 6 Temporal containment

that has a causal relationship with any process of $\{q_1, q_2, \ldots\}$ *has a causal relationship with* r, i.e.

$$\forall p.((\forall q \in \{q_1, q_2, \ldots\}.(C(p,q))). \Rightarrow C(p,r)) \tag{4}$$

Now taking the containment relation as fundamental, we can conceive of a non-Boolean model for the logic of spatiotemporal structures. Letting

$$\mathcal{I} \equiv \{I \in 2^{\mathbf{Proc}} \mid \forall r.(\forall p.(\forall q \in I.C(p,q) \Rightarrow C(p,r)) \Leftrightarrow r \in I)\} \tag{5}$$

be the collection of all time intervals, we stipulate that the time interval $[q_1, q_2, \ldots]$ be the minimum element (with respect to the set inclusion ordering) in $\mathcal{I}$ that contains the set $\{q_1, q_2, \ldots\}$. This definition of time interval indeed satisfies the condition (4). The existence of the minimum element is assured by the fact that $\mathcal{I}$ is closed under the set-theoretic intersection. Note that $\mathcal{I}$ is *not* closed under the set-theoretic union.

Example 5 In Fig. 7, we have $\mathcal{I} = \{\phi$, $\{p_1\}$, $\{p_2\}$, $\{p_1, p_2\}$, $\{p_3\}$, $\{p_1, p_3\}$, $\{p_4\}$, $\{p_1, p_4\}$, $\{p_2, p_4\}$, $\{p_1, p_2, p_4\}$, $\{p_3, p_4\}$, $\{p_1, p_3, p_4\}$, $\{q_1\}$, $\{q_2\}$, $\{p_1, q_1, q_2\}$, $\{q_3\}$, $\{p_2, p_3, q_3\}$, $\{q_1, q_3\}$, $\{q_2, q_3\}$, $\{p_1, q_1, q_2, q_3\}$, $\{p_1, p_2, p_3, q_1, q_2, q_3\}$, $\{q_4\}$, $\{q_1, q_4\}$, $\{q_2, q_4\}$, $\{p_1, q_1, q_2, q_4\}$, $\{q_3, q_4\}$, $\{q_1, q_3, q_4\}$, $\{q_5\}$, $\{q_1, q_5\}$, $\{q_2, q_5\}$, $\{p_1, q_1, q_2, q_5\}$, $\{q_3, q_5\}$, $\{q_1, q_3, q_5\}$, $\{q_2, q_3, q_5\}$, $\{p_1, q_1, q_2, q_3, q_5\}$, $\{p_4, q_4, q_5\}$, $\{p_4, q_1, q_4, q_5\}$, $\{p_4, q_2, q_4, q_5\}$, $\{p_1, p_4, q_1, q_2, q_4, q_5\}$, $\{p_4, q_3, q_4, q_5\}$, $\{p_2, p_3, p_4, q_3, q_4, q_5\}$, $\{p_4, q_1, q_3, q_4, q_5\}$, $\{q_1, r_1\}$, $\{r_2\}$, $\{q_2, q_3, q_4, r_2\}$, $\{q_1, r_1, r_2\}$, $\{p_1, q_1, q_2, q_3, q_4, r_1, r_2\}$, $\{q_5, r_3\}$, $\{q_1, q_5, r_1, r_3\}$, $\{q_5, r_2, r_3\}$, $\mathbf{Proc}\}$.

As in the non-relativistic case, we obtain a logic from time structure by considering the time interval of a process as the truth value of an atomic proposition. Now the resulting algebra is *ortholattice*, which is not Boolean in general. The associated logic is called *orthologic* or *minimal quantum logic* (for the details of orthologic, see [2] and [3]):

1. For any proposition p, $\neg p$ denotes the proposition that the proposition p is not true. The truth value of $\neg p$ is defined as $[\neg p] \equiv \{q \mid \forall r \in [p].C(q,r)\}$ (orthocomplement relative to $\mathcal{I}$), which amounts to the time interval that p is not true. The following facts follow from the definition.

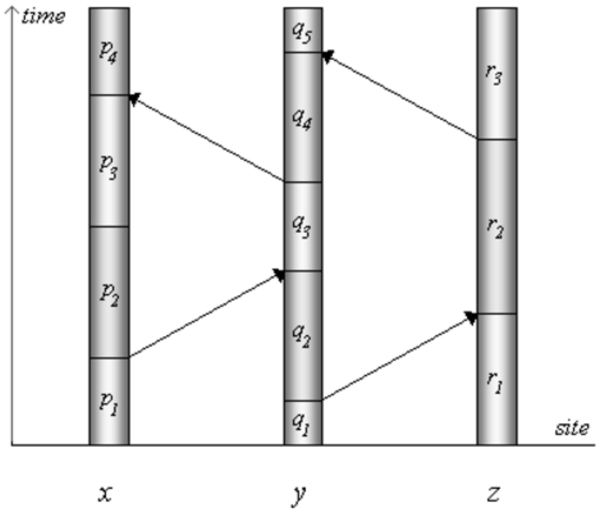


Fig. 7 A counterexample to distributivity

- $[\neg p] \in \mathcal{I}$.
- $[\neg\neg p] = [p]$.
- $[p] \subseteq [q]$ if and only if $[\neg q] \subseteq [\neg p]$.

2. For any propositions p and q, $p \wedge q$ denotes the proposition that the propositions p and q are both true. The truth value of $p \wedge q$ is defined as $[p \wedge q] \equiv [p] \cap [q]$ (set-theoretic intersection), which amounts to the time interval that p and q are both true. Since $\mathcal{I}$ is closed under the set-theoretic intersection, $\wedge$ corresponds to the infimum operator on $\mathcal{I}$ with respect to the set inclusion ordering.
3. For any propositions p and q, $p \vee q$ denotes the proposition that at least one of the propositions p and q is true. The truth value of $p \vee q$ is defined as $[p \vee q] \equiv [\neg(\neg p \wedge \neg q)]$, which amounts to the time interval that at least one of p and q is true. Since $\neg$ has the above-mentioned properties and $\wedge$ is the infimum operator on $\mathcal{I}$, $\vee$ corresponds to the supremum operator on $\mathcal{I}$ with respect to the set inclusion ordering. Note that we have $[p_1, p_2, \ldots] = [p_1 \vee p_2 \vee \ldots]$ for atomic propositions $p_1, p_2, \ldots$.

Example 6 A typical statement which is always true in Boolean logic but not necessarily true in orthologic is the distributive law of $\wedge$ over $\vee$, i.e.

$$[(p \vee q) \wedge r] = [(p \wedge r) \vee (q \wedge r)]. \tag{6}$$

In Fig. 7, we can find a counterexample to distributivity: since $[p_2 \vee p_3] = \{p_2, p_3, q_3\}$ and $[q_3] = \{q_3\}$, we infer $[(p_2 \vee p_3) \wedge q_3] = \{q_3\}$, while since $[p_2 \wedge q_3] = \phi$ and $[p_3 \wedge q_3] = \phi$, we infer $[(p_2 \wedge q_3) \vee (p_3 \wedge q_3)] = \phi$. The failure of distributivity illustrates the fact that the analysis of spatiotemporal structures deduces non-Boolean orthologic when global synchronized clocks are not available.

4 Conclusion and Future Work

For further study in logic and mathematics, we present here a more formal treatment of our approach:

Proc: the set of all processes from the past to the future. One may think of it as a set of names assigned to the states of a system. Mathematically, it is simply a nonempty countable set.

C: the causality relation. Physically, the relation between two processes holds if and only if some material or electrical information can be transmitted from one of the processes to the other. Intuitively, the two processes are causally related if and only if it can be confirmed that one of the processes ends before the other begins (of course, it is not a rigorous definition in that it commits the fallacy of assuming what one is trying to define—*time*). Mathematically, it is a irreflexive, symmetric binary relation on **Proc**.

S: the simultaneity relation. Physically, the relation between two processes holds if and only if there is no causal relationship. Mathematically, it is the complement relation of C relative to **Proc**.

Specifying **Proc** and C (or S) corresponds to determining the spacetime structure of a particular situation.

Given atomic processes p and q, we can think of the complex processes $\neg p$ and $p \wedge q$ that intuitively mean the state in which p is not occurring, and the state in which both p and q are occurring, respectively. $p \vee q$ is defined as an abbreviation for $\neg(\neg p \wedge \neg q)$. We begin with atomic processes, then inductively build up complex processes using the logical connectives $\neg$ and $\wedge$. Now we redefine **Proc** as the set of all atomic and complex processes.

The next step is to introduce a logico-algebraic structure of spacetime by using **Proc** and C (or S). By (5), an element I of the spacetime is a set of processes satisfying:

$$r \in I \Leftrightarrow \forall p.(\forall q.(C(p,q) \Rightarrow C(p,r)). \tag{7}$$

This expression is equivalent to:

$$r \in I \Leftrightarrow \forall p.(\forall q.(q \in I \Rightarrow C(p,q)) \Rightarrow C(p,r)) \tag{8}$$

or:

$$r \in I \Leftrightarrow \forall p.(S(p,r) \Rightarrow \exists q \in I \text{ s.t. } (S(p,q) \text{ and } q \in I)). \tag{9}$$

Here we have used the contraposition of implication. The last expression shows that I has exactly the same mathematical structure as a proposition of orthoframe (for the details of orthoframe and orthologic, see [3]).

We denote the totality of such I by $\mathcal{I}$. Then a semantic function $[\cdot]$ is defined as a map from **Proc** to $\mathcal{I}$ satisfying:

– For atomic process p, $[p]$ is the smallest element of $\mathcal{I}$ that contains p.
– $[\neg p] \equiv \{q \in \textbf{Proc} | \forall r \in \textbf{Proc}.(S(q,r) \Rightarrow r \notin [p])\}$.
– $[p \wedge q] \equiv [p] \cap [q]$ (set-theoretic intersection).

As we have seen, the resulting logical structure matches that of orthologic, which lacks the distributivity law over logical conjunction and disjunction. This restriction inhibits spatially separated processes from having the common clock or synchronized clocks. In other words, the *conventionality of simultaneity* claim amounts to the assertion that logico-algebraic structure of spacetime is a non-distributive one.

Based on this foundation, further study could include the development of modal logics. Modal operators are employed to describe the time-dependent properties such as mutual exclusion, accessibility, and correctness [5]. It is interesting to see what future research will reveal about temporal modal logic when the base logic (usually Boolean) is replaced by orthologic.

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